UNSTEADY MOTION OF A FLUID NEAR A DISK ROTATING IN A MAGNETIC FIELD

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The effect of a magnetic field on the velocity distribution in a fluid close to an unsteadily rotating disk is investigated.

Rozin [1] investigated the unsteady laminar boundary layer near a rotating unbounded disk for particular laws of variation of the circular velocity: $\omega(t) = \omega_0 t^n$ and $\omega(t) = \omega_0 e^{\xi t}$.

Following Rozin's method, we will calculate the unsteady motion of an incompressible, electrically conducting, viscous fluid near a similar unbounded disk in the presence of a constant transverse magnetic field. For the radial velocity of the external potential flow we choose the law $u_0 = ar(e^{\Omega t})^m$, and the space-time relationship for the circular velocity of the disk has the form $v_0 = \Omega r(e^{\Omega t})^n$. The calculations show that the magnetic field has an appreciable effect on the velocity profile in the boundary layer.

The problem of steady motion was solved in [2] by the Karman-Pohlhausen integral method.

EQUATIONS OF MOTION

Let a disk of infinite radius, previously at rest, begin to rotate in its own plane (z = 0) around the axis r = 0 with variable angular velocity. We assume that the fluid is incompressible, viscous, and electrically conducting and occupies the region z > 0. The applied magnetic field is directed perpendicular to the plane of the disk and is uniform and constant. The magnetic Prandtl number is assumed to be so small that electric currents in the fluid have no effect on the applied magnetic field.

In cylindrical coordinates the Navier-Stokes equations for unsteady motion in the presence of a magnetic field have the form [3]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right] - \frac{\sigma B_0^2}{\rho}u,$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right] - \frac{\sigma B_0^2}{\rho}v,$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right],$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \qquad (1)$$

On the assumption that

$$u_0 = ar(e^{\Omega t})^m \text{ and } v_0 = \Omega r(e^{\Omega t})^n, \qquad (2)$$

the initial and boundary conditions of system (1) will be

$$u = v = w = 0 \text{ for } t \le 0 \text{ and for any } z,$$

$$\left.\begin{aligned} &u = w = 0,\ v = v_0 = \Omega r(e^{\Omega t})^n \text{ for } z = 0 \\ &u = u_0 = ar(e^{\Omega t})^m,\quad v = 0 \quad \text{for } z \to \infty \end{aligned}\right\}\text{and for any } z. \qquad (3)$$

The pressure distribution when $z \to \infty$ will be determined from the condition

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial r} + \frac{\sigma B_0^2}{\rho}u_0 = ar(e^{\Omega t})^m\left[\Omega m + a(e^{\Omega t})^m + \frac{\sigma B_0^2}{\rho}\right]. \qquad (4)$$

We introduce the following dimensionless quantities:

$$u = \Omega r U(Z, T), \quad v = \Omega r V(Z, T),$$

$$w = \sqrt{\nu\Omega}W(Z, T), \quad p = -\frac{\rho}{2}ar^2 e^{mT} \times \left[\Omega m + ae^{mT} + \frac{\sigma B_0^2}{\rho}\right] + \rho\nu\Omega P(Z, T),$$

$$r = \sqrt{\nu/\Omega}R, \quad z = \sqrt{\nu/\Omega}Z, \quad t = (1/\Omega)T,$$

$$k = a/\Omega, \quad \lambda = \sigma B_0^2/\rho\Omega. \qquad (5)$$

Substituting (5) in Eq. (1), we obtain the system

$$\frac{\partial U}{\partial T} + W\frac{\partial U}{\partial Z} + U^2 - V^2 = ke^{mT}[m + \lambda + ke^{mT}] + \frac{\partial^2 U}{\partial Z^2} - \lambda U,$$

$$\frac{\partial V}{\partial T} + W\frac{\partial V}{\partial Z} + 2UV = \frac{\partial^2 V}{\partial Z^2} - \lambda V,$$

$$\frac{\partial W}{\partial T} + W\frac{\partial W}{\partial Z} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial Z^2},$$

$$2U + \frac{\partial W}{\partial Z} = 0, \qquad (6)$$

with boundary conditions

$$U = W = 0,\quad V = e^{nT} \qquad \text{for } Z = 0,$$

$$U = ke^{mT},\quad V = 0 \qquad \text{for } Z \to \infty. \qquad (7)$$

METHOD OF SOLUTION

To integrate system (6) with boundary conditions (7) we seek solutions of the form [1,4]:

$$U = e^{mT}[f_0'(\eta) + f_1'(\eta)e^{mT} + \cdots],$$
$$V = e^{nT}[g_0(\eta) + g_1(\eta)e^{mT} + \cdots],$$
$$W = -4e^{mT}[f_0(\eta) + f_1(\eta)e^{mT} + \cdots],$$
$$P = 2e^{mT}[h_0(\eta) + h_1(\eta)e^{mT} + \cdots], \qquad (8)$$

where $\eta = Z/2$. Functions $f_i(\eta)$, $f_i'(\eta)$, and $g_i(\eta)$ satisfy the boundary conditions

$$\left.\begin{aligned} f_0(\eta) = f_0'(\eta) = f_i(\eta) = f_i'(\eta) = 0 \\ g_0(\eta) = 1, \quad g_i(\eta) = 0 \end{aligned}\right\} \text{ for } \eta = 0,$$
$$\left.\begin{aligned} f_0'(\eta) = k, \quad f_i'(\eta) = 0 \\ g_0(\eta) = g_i(\eta) = 0 \end{aligned}\right\} \text{ for } \eta \to \infty, \qquad (9)$$

where $i = 1, 2, 3, \ldots$.

From (6) and (8) we obtain the system of equations

$$[mf_0' + 2mf_1'e^{mT} + \cdots] + e^{mT}[f_0'^2 - 2f_0f_0'' + \cdots] -$$
$$- e^{(2n-m)T}[g_0^2 + 2g_0g_1e^{mT} + \cdots] = k(m+\lambda) +$$
$$+ k^2e^{mT} + \frac{1}{4}[f_0''' + f_1'''e^{mT} + \cdots] - \lambda[f_0' + f_1'e^{mT} + \cdots],$$
$$[ng_0 + (m+n)g_1e^{mT} + \cdots] + 2e^{mT}[f_0'g_0 - f_0g_0' + \cdots] =$$
$$= \frac{1}{4}[g_0'' + g_1''e^{mT} + \cdots] - \lambda[g_0 + g_1e^{mT} + \cdots],$$
$$[mf_0 + 2mf_1e^{mT} + \cdots] - 2e^{mT}[f_0f_0' + \cdots] =$$
$$= \frac{1}{4}[h_0' + f_0'' + (h_1' + f_1'')e^{mT} + \cdots]. \qquad (10)$$

System (10) can be solved if $2n - m$ is divisible by m.

SOLUTION FOR CASE n = m

Substituting $n = m$ in Eq. (10) and equating the coefficients of equal powers of e^{mT}, we obtain the following system of equations for the first two approximations:

$$f_0''' - 4(m+\lambda)f_0' = -4k(m+\lambda),$$
$$g_0'' - 4(m+\lambda)g_0 = 0,$$
$$h_0' = -f_0'' + 4mf_0 \qquad (11)$$

and

$$f_1''' - 4(2m+\lambda)f_1' = -4k^2 + 4(f_0'^2 - 2f_0f_0'' - g_0^2),$$
$$g_1'' - 4(2m+\lambda)g_1 = 8(f_0'g_0 - f_0g_0'),$$
$$h_1' = -f_1'' + 8(mf_1 - f_0f_0') \qquad (12)$$

with boundary conditions

$$\left.\begin{aligned} f_0(\eta) = f_0'(\eta) = f_1(\eta) = f_1'(\eta) = 0 \\ g_0(\eta) = 1, \quad g_1(\eta) = 0 \end{aligned}\right\} \text{ for } \eta = 0,$$
$$\left.\begin{aligned} f_0'(\eta) = k, \quad f_1'(\eta) = 0 \\ g_0(\eta) = g_1(\eta) = 0 \end{aligned}\right\} \text{ for } \eta \to \infty. \qquad (13)$$

The solutions of Eqs. (11) and (12) are

$$f_0'(\eta) = k[1 - e^{-2\eta\sqrt{m+\lambda}}],$$
$$f_0(\eta) = \frac{k}{2\sqrt{m+\lambda}}\left[e^{-2\eta\sqrt{m+\lambda}} - 1\right] + k\eta,$$
$$g_0(\eta) = e^{-2\eta\sqrt{m+\lambda}},$$
$$h_0(\eta) - h_0(0) = \frac{k\lambda}{m+\lambda}\left[e^{-2\eta\sqrt{m+\lambda}} - 1\right] +$$
$$+ 2k\eta\left[\eta - \frac{1}{\sqrt{m+\lambda}}\right], \qquad (14)$$

and

$$f_1'(\eta) = \frac{m^2 + k^2(9m^2 + 20m\lambda + 12\lambda^2)}{m^2(2m+3\lambda)} \times$$
$$\times e^{-2\eta\sqrt{2m+\lambda}} - \frac{1+k^2}{2m+3\lambda}e^{-4\eta\sqrt{m+\lambda}} +$$
$$+ \frac{4k^2}{m^2}(m\eta\sqrt{m+\lambda} - m - \lambda)e^{-2\eta\sqrt{m+\lambda}},$$
$$f_1(\eta) = -\frac{m^2 + k^2(9m^2 + 20m\lambda + 12\lambda^2)}{2m^2(2m+3\lambda)\sqrt{2m+\lambda}} \times$$
$$\times e^{-2\eta\sqrt{2m+\lambda}} + \frac{1+k^2}{4(2m+3\lambda)\sqrt{m+\lambda}}e^{-4\eta\sqrt{m+\lambda}} +$$
$$+ \frac{k^2}{m^2}\left(\frac{m+2\lambda}{\sqrt{m+\lambda}} - 2m\eta\right)e^{-2\eta\sqrt{m+\lambda}} +$$
$$+ \{[2m^2 + 2k^2(9m^2 + 20m\lambda + 12\lambda^2)]\sqrt{m+\lambda} -$$
$$- [m^2 + k^2(9m^2 + 28m\lambda + 24\lambda^2)]\sqrt{2m+\lambda}\} \times$$
$$\times [4m^2(2m+3\lambda)\sqrt{(m+\lambda)(2m+\lambda)}]^{-1},$$
$$g_1(\eta) = \frac{4k}{m}[(m + \lambda - m\eta\sqrt{m+\lambda})e^{-2\eta\sqrt{m+\lambda}} -$$
$$- (m+\lambda)e^{-2\eta\sqrt{2m+\lambda}}]. \qquad (15)$$

In the absence of the magnetic field ($\lambda = 0$) and for $m = 1$ and $k = 0$ we obtain Rozin's result [1] from relationships (14) and (15).

The velocity profiles are shown in Figs. 1 and 2. With increase in the magnetic field the radial velocity increases, whereas the circular velocity decreases. This result agrees with the data of [2] for the case of steady motion of a fluid near a rotating disk in the presence of a transverse constant magnetic field.

SOLUTION FOR CASE m = 2n

Substituting $m = 2n$ in Eq. (10) and equating the coefficients of equal powers of e^{2nT}, we obtain the following system of equations for the first approximation:

$$f_0''' - 4(2n+\lambda)f_0' = -4k(2n+\lambda) - 4g_0^2,$$
$$g_0'' - 4(2n+\lambda)g_0 = 0,$$
$$h_0' = -f_0'' + 8nf_0. \qquad (16)$$

Its solution for $k = 0$ will be:

$$f_0'(\eta) = \frac{1}{2n+3\lambda}\left[e^{-2\eta\sqrt{2n+\lambda}} - e^{-4\eta\sqrt{n+\lambda}}\right],$$
$$f_0(\eta) = \frac{1}{2(2n+3\lambda)}\left[\frac{1}{2\sqrt{n+\lambda}}e^{-4\eta\sqrt{n+\lambda}} -\right.$$
$$\left. - \frac{1}{\sqrt{2n+\lambda}}e^{-2\eta\sqrt{2n+\lambda}}\right] + \frac{1}{2(2n+3\lambda)} \times \qquad (17)$$

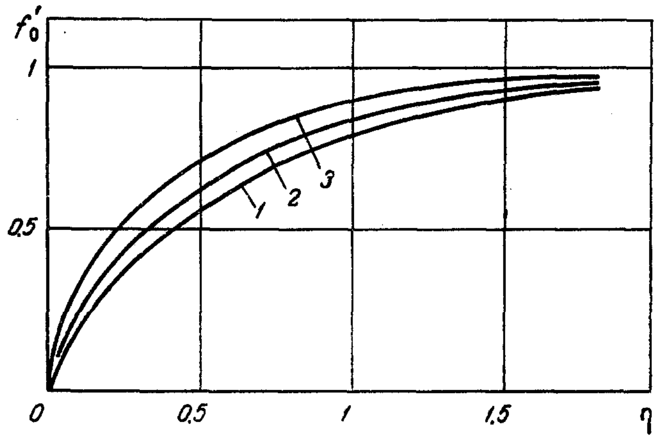

Fig. 1. Distribution of radial velocity for k = 1: 1) λ = 0; 2) λ = 1/2; 3) λ = 1.

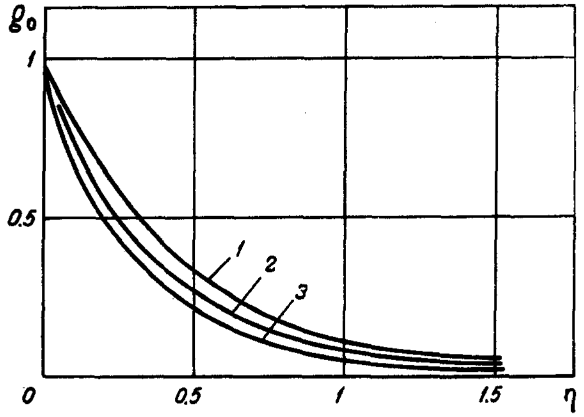

Fig. 2. Distribution of circular velocity for k = 1: 1) λ = 0; 2) λ = 1/2; 3) λ = 1.

$$\times\left[\frac{1}{\sqrt{2n+\lambda}}-\frac{1}{2\sqrt{n+\lambda}}\right],$$

$$g_0(\eta)=e^{-2\eta\sqrt{2n+\lambda}},$$

$$h_0(\eta)-h_0(0)=\frac{1}{2n+3\lambda}\left[\frac{n+3\lambda}{2(n+\lambda)}e^{-4\eta\sqrt{n+\lambda}}-\right.$$

$$-\frac{\lambda}{2n+\lambda}e^{-2\eta\sqrt{2n+\lambda}}+4n\left(\frac{1}{\sqrt{2n+\lambda}}-\right.$$

$$\left.\left.-\frac{1}{2\sqrt{n+\lambda}}\right)\eta\right]-\frac{n}{2(2n+\lambda)(n+\lambda)}. \qquad (17)\ \text{(cont'd)}$$

For $\lambda = 0$ and for $n = 1$ formulas (17) are identical with the relationships obtained by Rozin [1].

NOTATION

r, θ, and z are the coordinates in the radial, circular, and axial directions; t is the time; u, v, and w are the radial, circular, and axial velocity components; u_0 is the radial velocity of external potential flux; v_0 is the circular velocity of the disk; $\omega(t)$ is the angular velocity of the disk; p is the pressure; ρ is the density; ν is the kinematic viscosity; B_0 is the characteristic of the applied magnetic field; σ is the electrical conductivity of fluid; R and Z are dimensionless coordinates in the radial and axial directions; $\eta = Z/2$ is a dimensionless coordinate; T is dimensionless time; U, V, and W are the radial, circular, and axial components of dimensionless velocity; P is dimensionless pressure; a, Ω, and ω_0 are constants with dimensionality t^{-1}; m, n, and ζ are positive numbers; $k = a/\Omega$ is a constant; $\lambda = \sigma B_0^2/\rho\Omega$ is the parameter characterizing the magnetic field.

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